GPS
Theory and Applications

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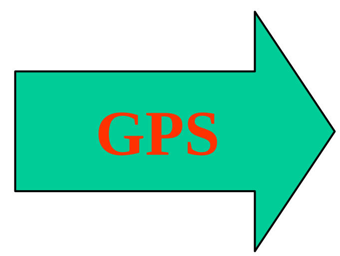
16.982

Fall 2002

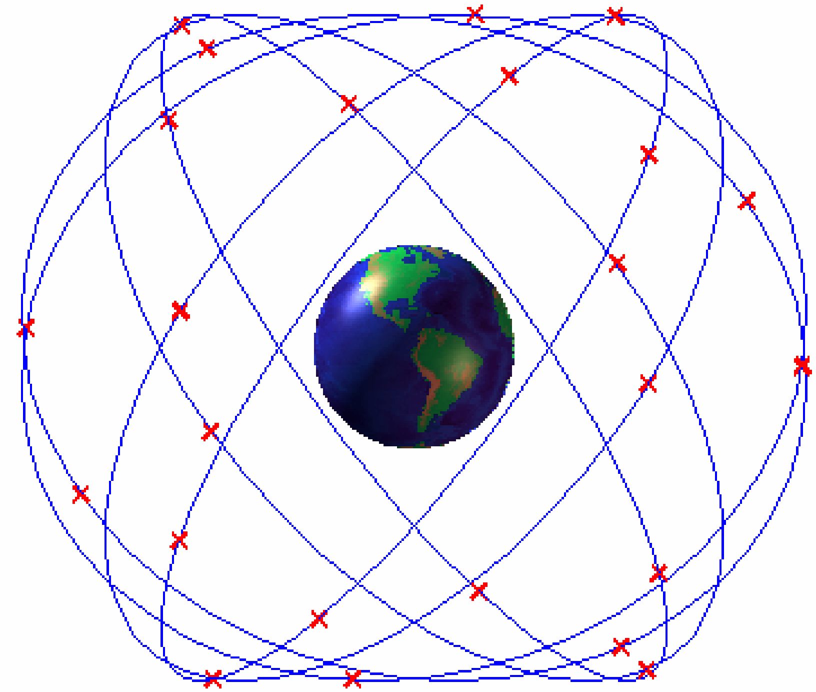
Background

- **Time transfer** is the key to determining your position
  - TRANSIT system developed in the 60's on the following premise:
    - Doppler shift in a signal broadcast from a spacecraft can be used to determine exact time of closest approach of the spacecraft
    - If satellite ephemeredes known, then your location precisely known **anywhere in the world**
- Problems:
  - Limited satellite coverage
  - Low accuracy

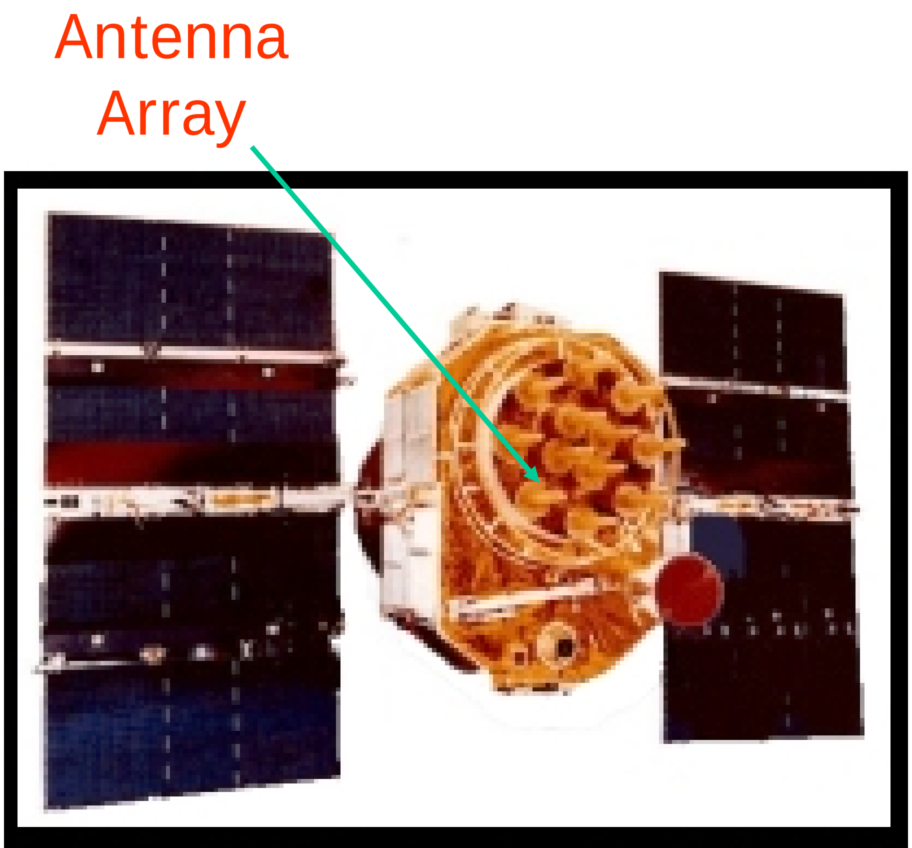


Navstar Constellation

- 24 total satellites with 3 spare
- 4 satellites in six orbital planes
  - 55º Inclination angle
  - 10,980 n.mi. altitude
  - 12 hour period
  - Nominally 6 satellites visible
- Currently 28 Block II/IIA/IIR on-orbit



Navstar Satellite

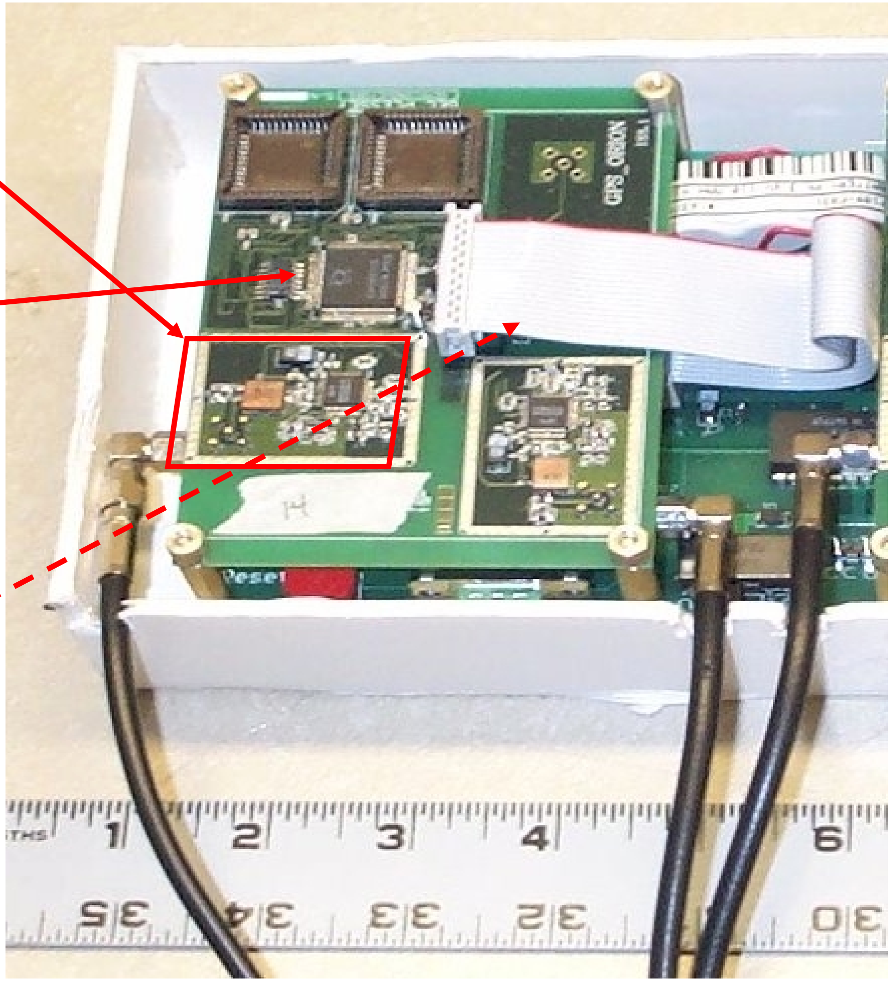


Block IIA Satellite

- **Block I** – Initial evaluation
  - 845 kg / 4.5 year design life
  - Launched 1978 – 85
- **Block II** – 63º to 55º inclination
  - Weight ~ 1500 kg / 7.5yrs
  - Restricted signals
- **Block IIA** – Advanced satellites (minor improvements)
- **Block IIR** – "Replenishment"
  - 2000 kg / 7.8 year life
  - Designed to operate for 14 days without ground contact
  - Can range and cross-link between themselves

GPS Receiver Hardware

- RF Front end section
  - Amplifies / filters / down converts GPS signal
- 12 Channel correlator
  - Slides a code replica to match received code phase
- Arm60 Processor
  - Closes tracking loops
  - Computes navigation solution.

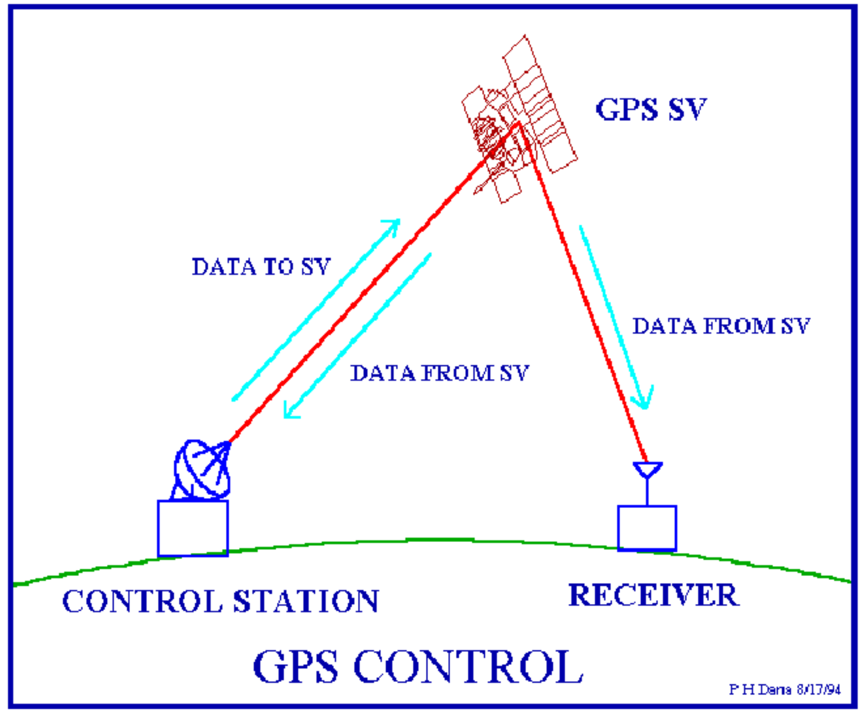


Information

- B. W. Parkinson, T. Stansell, R. Beard, and K. Gromov, ``A History of Navigation,'' Navigation, J. Inst. Navigation, vol. 42, no. 1, pp. 109-164, 1995.
  - See class web page
- Good place to start by Peter H. Dana
  - www.colorado.edu/geography/gcraft/notes/gps/gps_f.html
  - And where most of these figures came from
- Thanks also to Professor Per Enge at Stanford University

Ground Segment

- Ground Input to:
  - Keep GPS time
  - Perform maneuvers
  - Check s/c performance
  - Maintain orbit (perform maneuvers)



Global Positioning System (GPS) Master Control and Monitor Station Network

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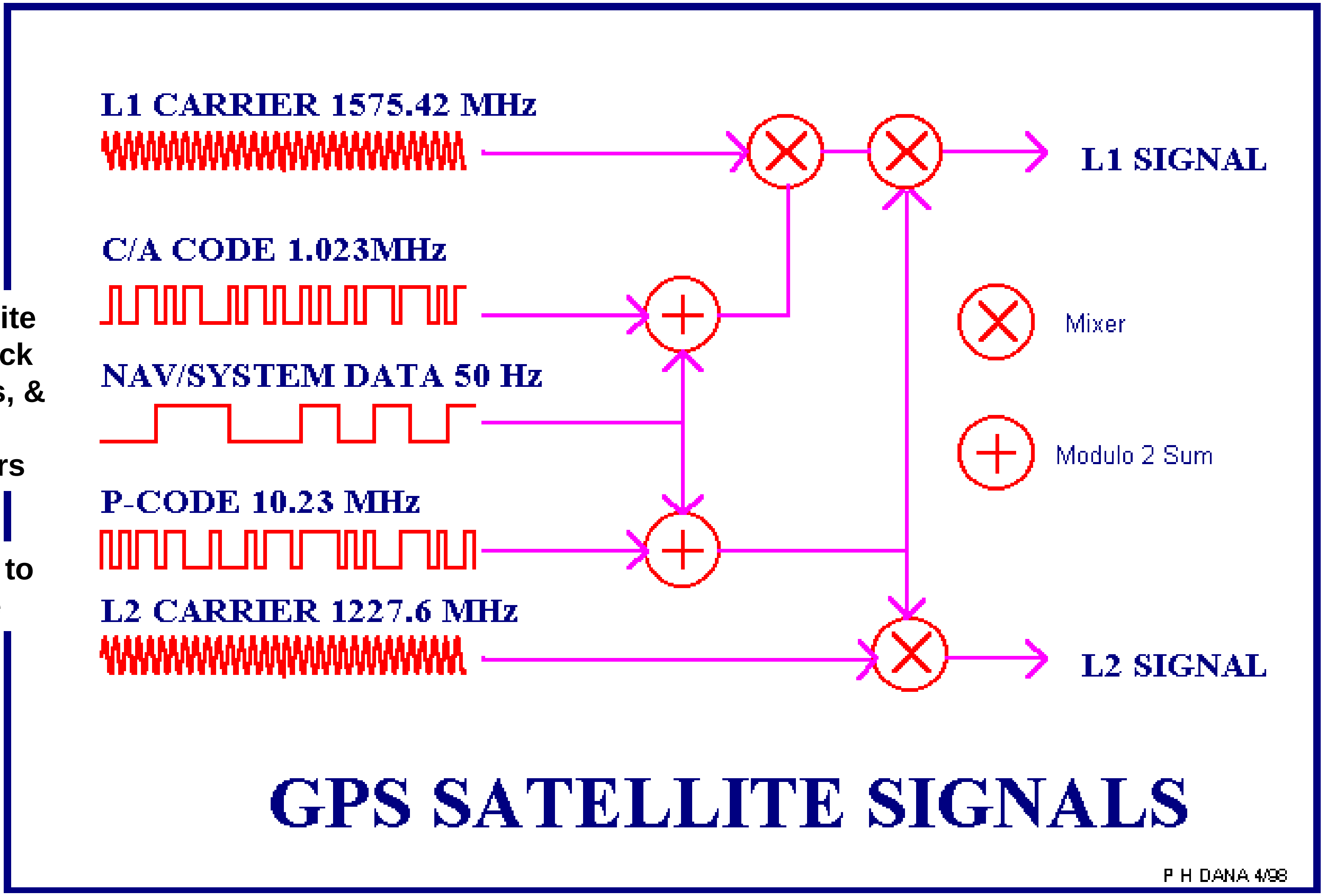
What is Transmitted?

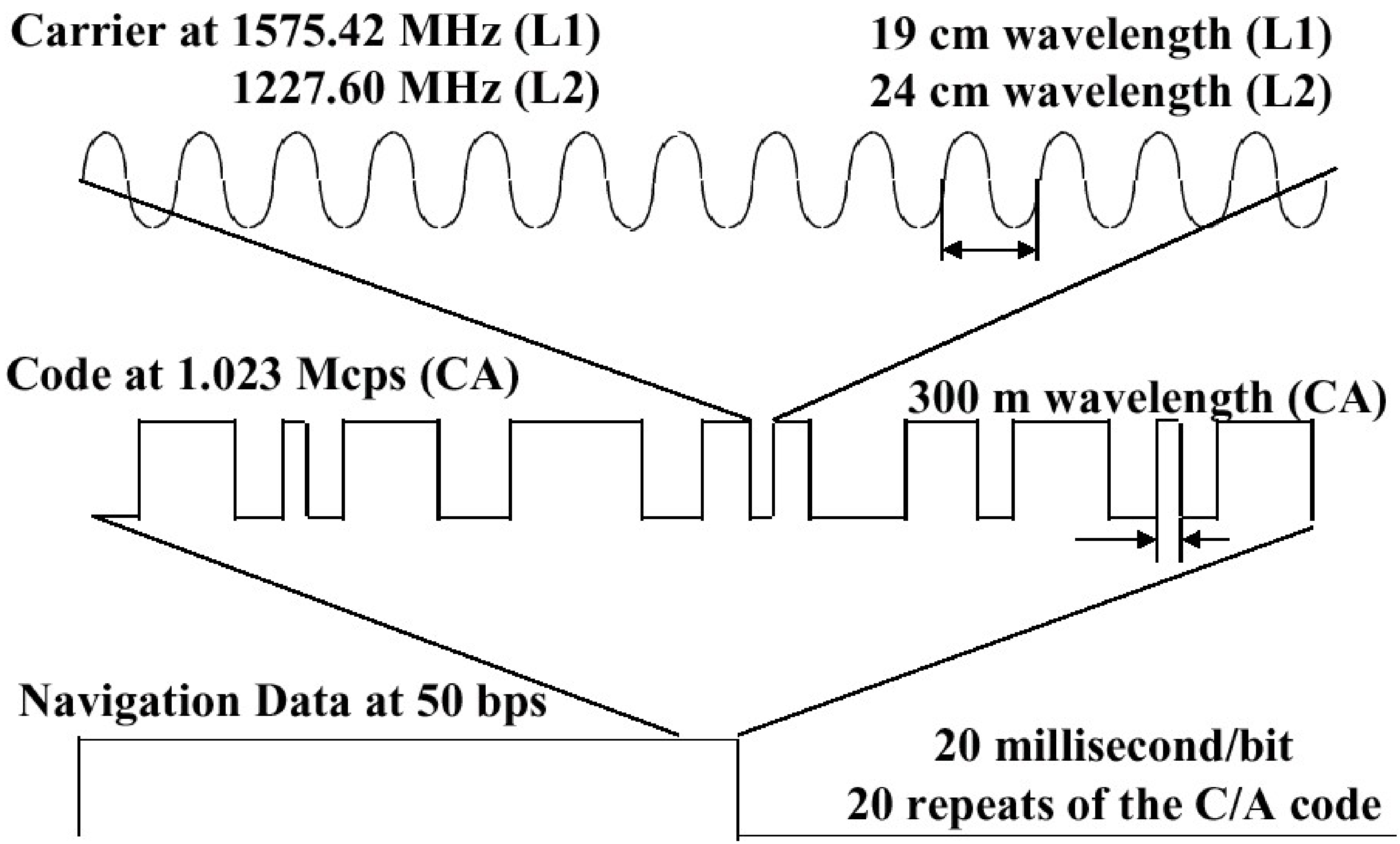
GPS satellite orbits, clock corrections, & system parameters

Encrypted to the Y code



C/A code on L_1 is the basis of civilian operations

GPS Signal Structure



GPS Signals (Code)

- C/A codes used to enable:
  - Accurate absolute range (10-100m) using signal propagation delay
  - Permits simultaneous measurements from several satellites (CDMA)

- Each satellite has a unique C/A code
  - One of a repeating sequence 1023 chips long
  - Rate of 1.023MHz (period of 1ms)
  - Appear random (pseudo random), but part of the deterministic “Gold Codes”
    - Selected because they provide very low cross-correlation

GPS Signals (Carrier)

- Carrier signal
  - Provides more accurate range measurements
    - Must overcome the ambiguity in the signal alignment
  - Provides very accurate Doppler measurements
- Dual frequency:
  - L1 (1575.42MHz) and L2 (1227.6MHz)
  - Compare signal propagation to develop accurate measure of ionosphere delay
    - Ionospheric group delay is strong function of frequency
  - Facilitates the carrier phase ambiguity resolution (wide-laning)

GPS Code Signal Acquisition

- Receiver replicates C/A (PRN) code to correlate with measured signal
  - Correlation performed to determine phase and time of code period

- Receiver "slides" a replica of the code in time until a strong correlation found with transmitted signal.
  - Wrong PRN code applied to an SV signal produces very low correlation.
  - Allows the receiver to pick up a very weak signal
  - Autocorrelation

$$(t) \quad \int_0^T c(t)c(t \quad)dt$$

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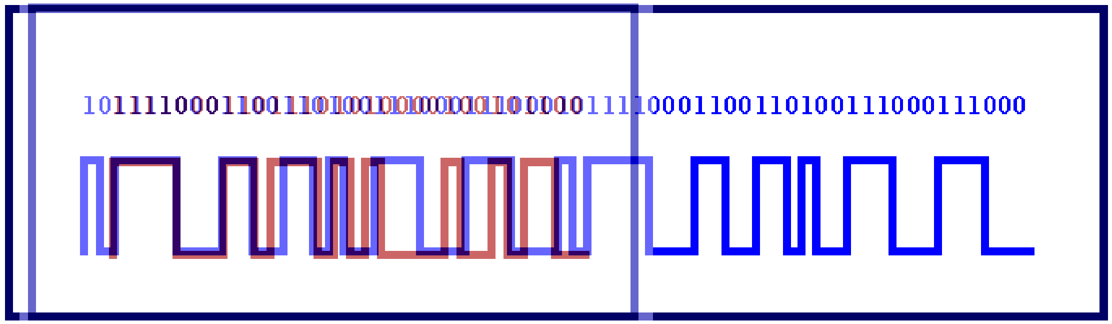
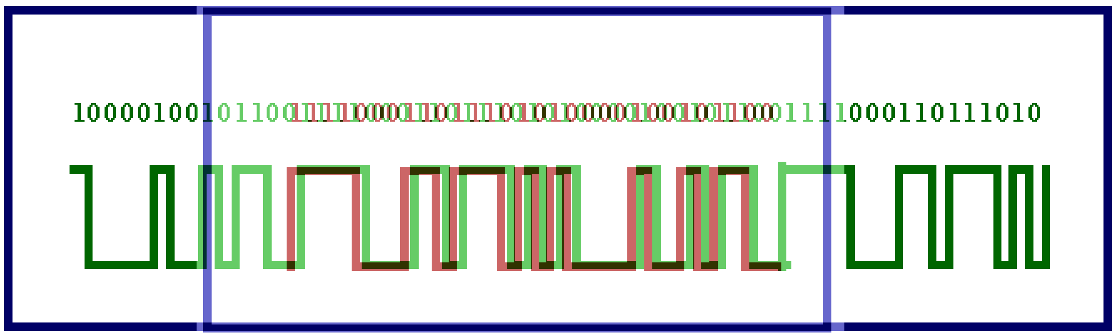
Correlation Visualization



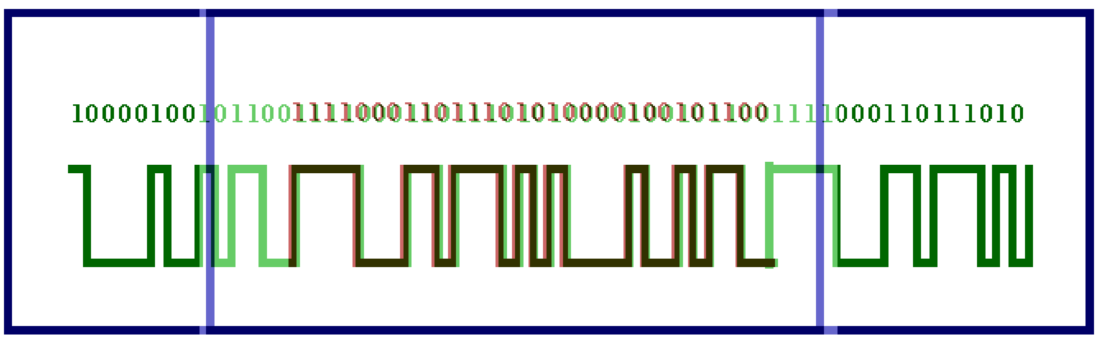
No Correlation with a Different PRN Code

Bad



Partial Correlation of Identical Receiver and Satellite PRN Codes

Better



Full Correlation (Code-Phase Lock) of Receiver and Satellite PRN Codes

Good

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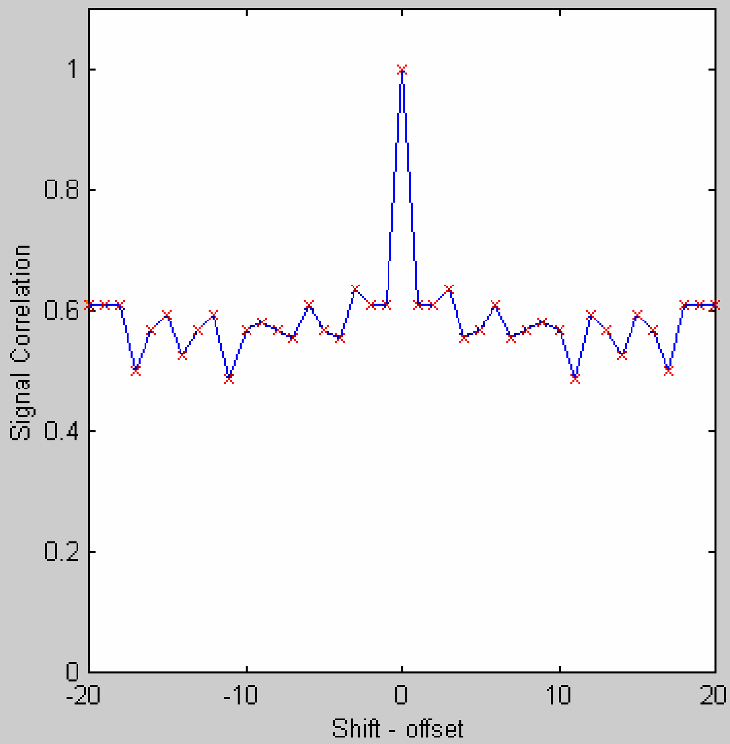
Correlation Movie



GPS Signal Acquisition Process

- Determine which satellites are visible
  - Approximate time/position, GPS almanac ➔ skyplot
- Determine approximate Doppler for each satellite
  - Reduces time-to-first-fix since defines frequency search space (estimate of receiver clock drift required)
- Must search **both** frequency and C/A code phase
  - Already discussed search of code phase
  - Large velocity of GPS satellites, so the received signals can have a large Doppler shift (+/- 5 KHz) that can vary rapidly
  - Develop frequency range estimate using Doppler shift
  - Search Doppler frequency range using 10-40 *frequency bins* of 500 Hz

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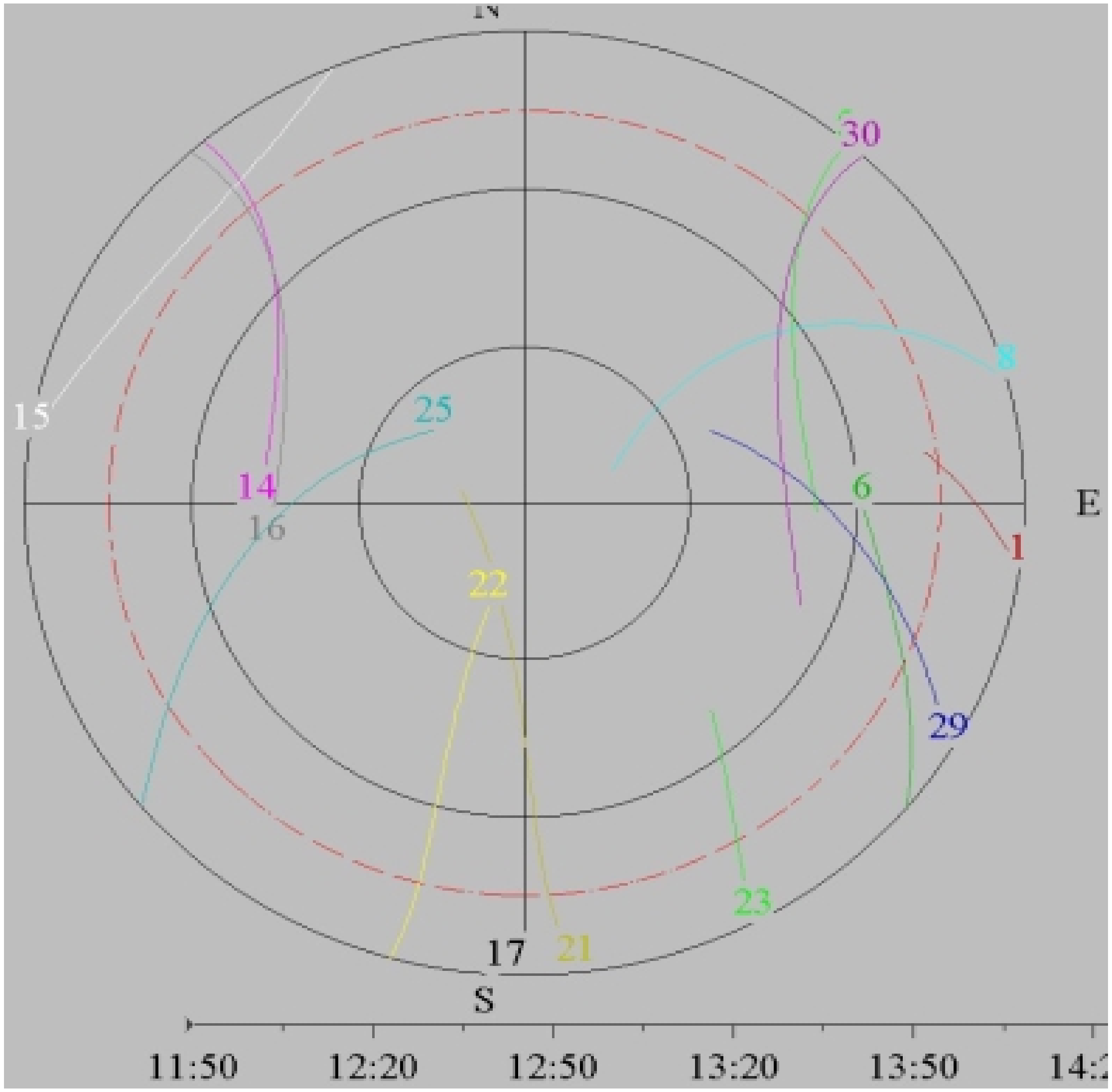
Typical Skyplot

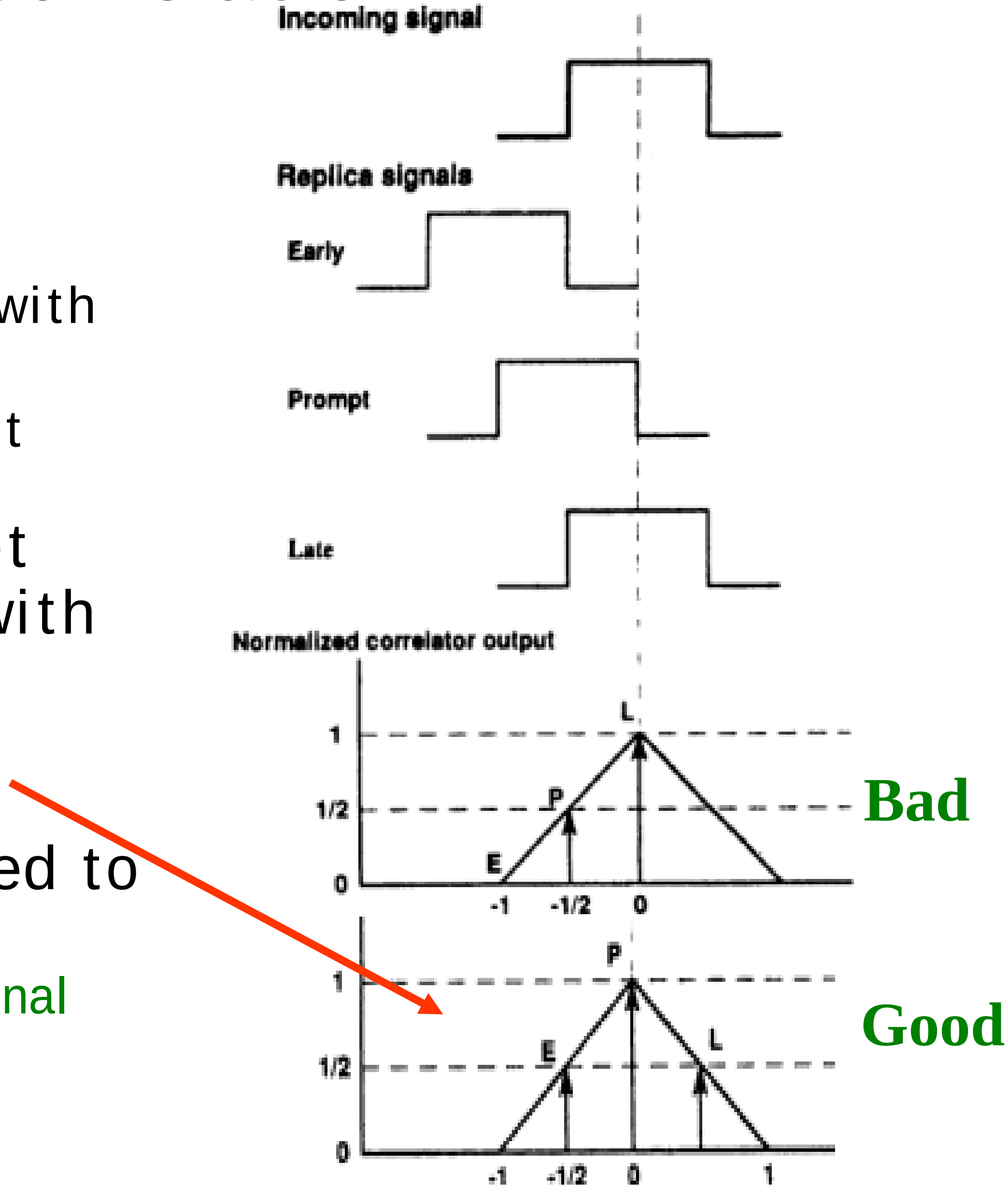


GPS Signal Acquisition Process

- Detect/confirm presence of signal
  - Ensure that not tracking a false signal
- Lock onto and track C/A code
  - Process of maintaining accurate alignment of the replica and received codes
  - Improves on our initial knowledge (aligned to within ½ chip and within the frequency bin)
  - Goal is to line up the replica and received codes as accurately as possible

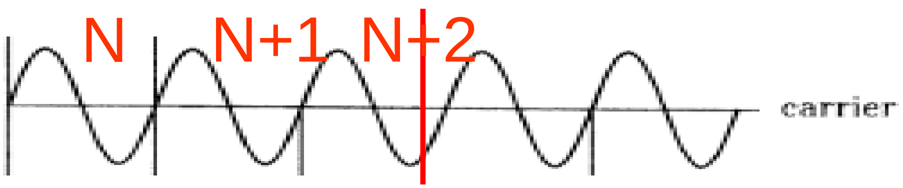
Correlation

- Generate 3 replicas separated by ½ chip
  - early / prompt / late
  - Correlate each replica with incoming signal
  - Slide replicas left/right
- **Desired result**: prompt replica synchronized with received signal
  $\Rightarrow$ autocorrelation peak
- Closed loop control used to hold replica at peak
  $\Rightarrow$ measure of phase of signal



GPS Signal Acquisition Process

- Lock onto and track Carrier
  - Use a PLL to track the phase of the carrier
  - "Carrier Doppler" used to supply frequency estimate to code tracking loop
    - Called *aiding*



- Carrier wavelength only ~19cm so this measurement is ambiguous
  - Very hard to resolve this ambiguity.
  - But when it is resolved, get a carrier phase range error ~ 2-5 mm (bias is ~10m!)
  - Bias can be removed if done **differentially**

Removing Carrier

- Before correlation can occur for acquisition, the signal must be translated to baseband
  - No carrier frequency
- Achieved by mixing with signals generated by a LO (Local Oscillator).

Data code

$$s_{L1}(t) \quad AD(t)C(t)\cos(2 \; f_{L1}t \qquad {}_{L1})$$

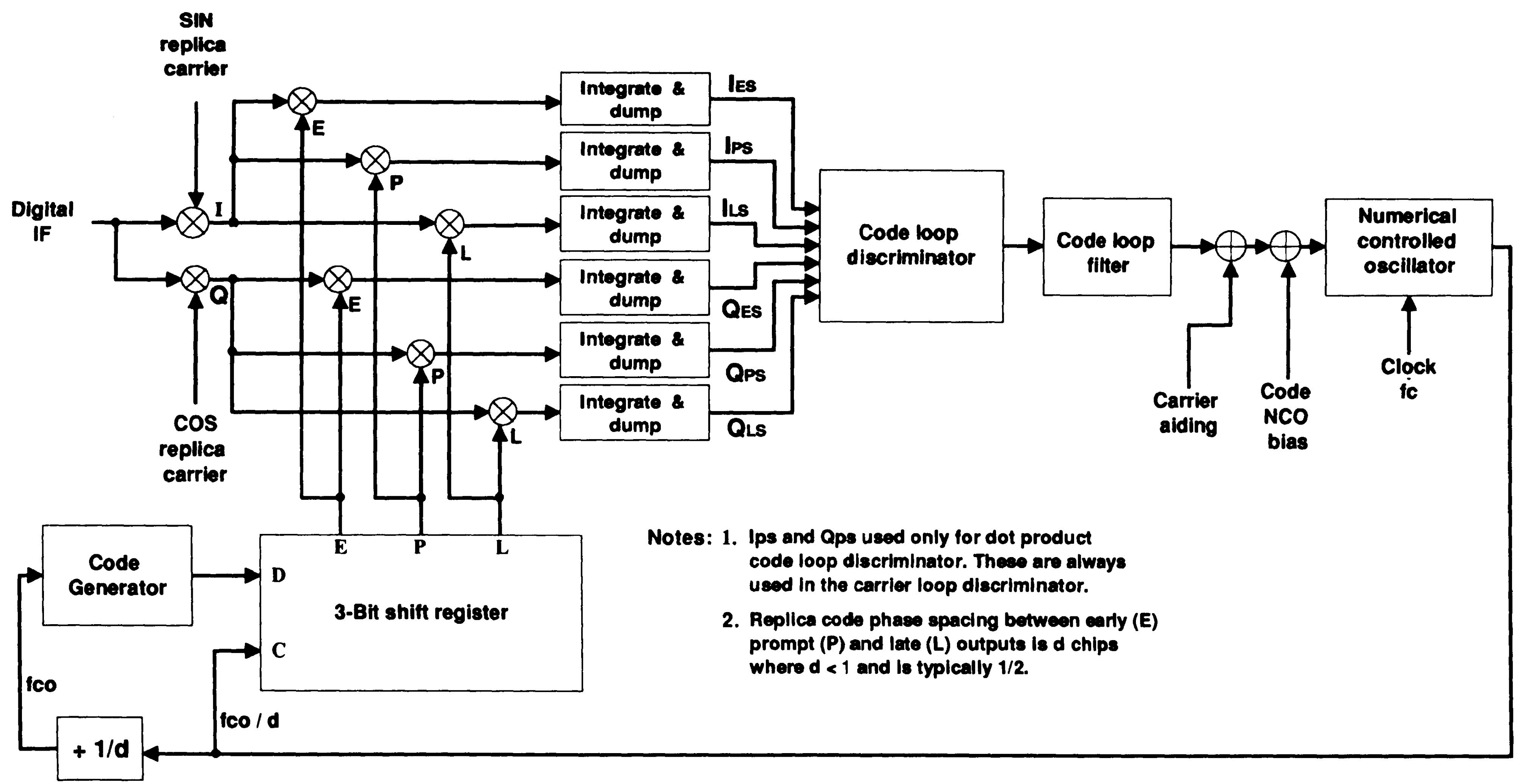
$$s_{\mathrm{LO}}(t) \quad \cos(2 \; f_{L1}t \qquad \hat{\ }_{L1})$$

$$s_{L1}(t) \quad s_{\mathrm{LO}}(t) \quad 0.5AD(t)C(t)\left[\cos(\; {}_{L1} \quad \hat{\ }_{L1}) \quad \cos(2 \; f_{L1}t \quad {}_{L1} \quad \hat{\ }_{L1})\right]$$

- After low-pass filter

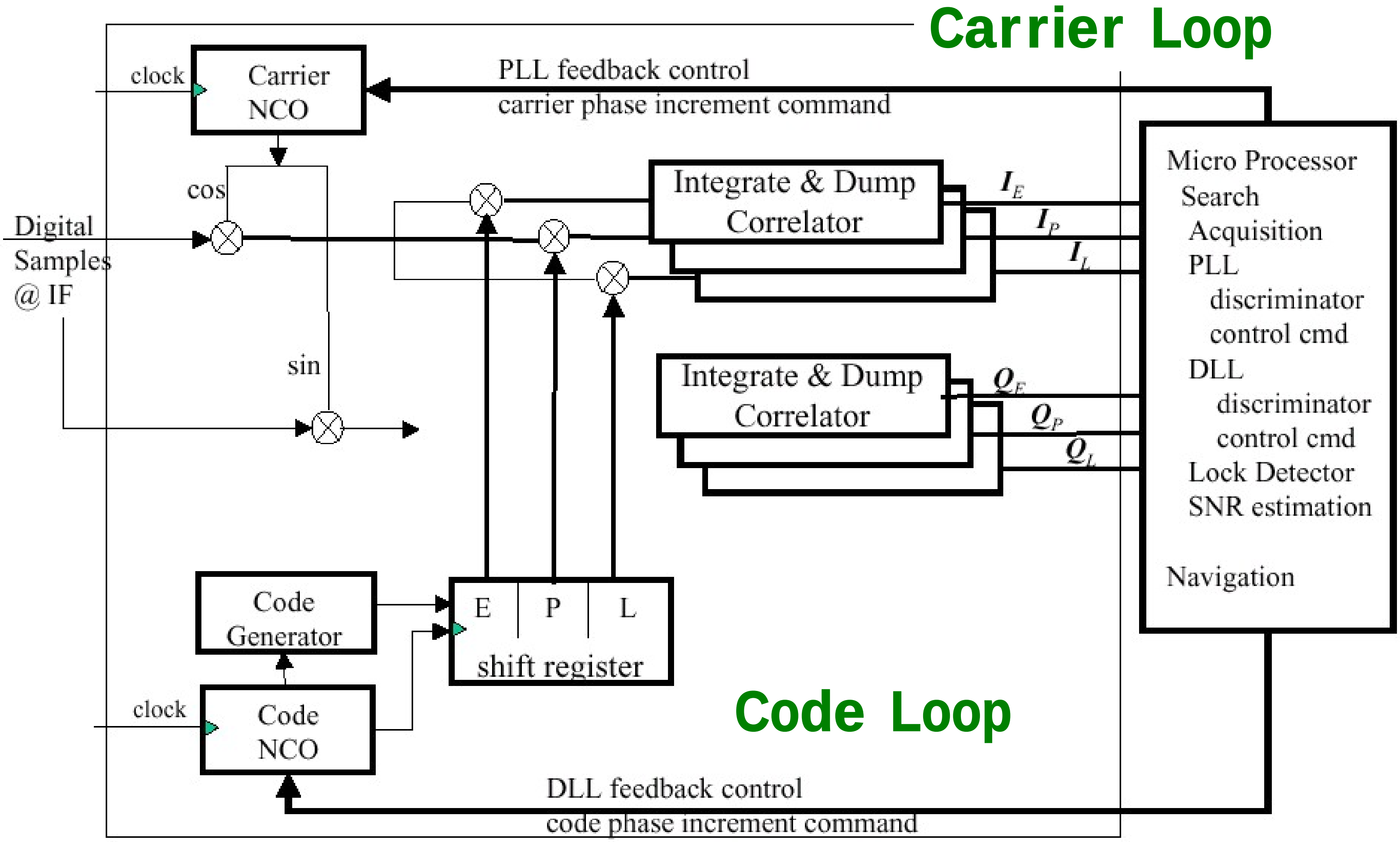
$$s_{L1}(t) \quad s_{\mathrm{LO}}(t) \quad 0.5AD(t)C(t)$$

Code Tracking Loop



Kaplan

GPS Receiver Layout



Measurements

- Output from code loop is an estimate of the "time of flight" of the signal
  - Difference between time of arrival and time sent (in data message)
  - Commonly converted to a pseudorange 'R' or 'τ'
  - C/A code repeats every millisecond
    - ~ 300km wavelength.
    - Satellites 20,000km away, so measurement is ambiguous, but it is easily resolved with a rough estimate of your location.

Code Point Positioning

- Write pseudorange as a function of
  - Spacecraft position X^k,...
  - Receiver position (ECEF) X_u,...
  - clock errors of s/c and receiver

$$ {}_u^k \quad \left[\sqrt{(x_u \quad x^k)^2 \quad (y_u \quad y^k)^2 \quad (z_u \quad z^k)^2} \quad b_u \quad B^k\right] \quad {}_u^k $$

  - Measure pseudorange from >= 4 satellites and you can solve for x_u, y_u, z_u, b_u

Stand alone Pseudoranging

- Pseudorange measurements

$$\tau_u^k = [r_u^k + b_u - B^k] + \nu_u^k$$

$$= \left[\sqrt{(x_u - x^k)^2 + (y_u - y^k)^2 + (z_u - z^k)^2} + b_u - B^k\right] + \nu_u^k$$

$$= f^k(x_u, y_u, z_u, b_u) + \nu_u^k$$

$$\bar{\tau} = \begin{bmatrix} f^1(x_u, y_u, z_u, b_u) \\ f^2(x_u, y_u, z_u, b_u) \\ \vdots \\ f^K(x_u, y_u, z_u, b_u) \end{bmatrix} + \bar{\nu}_u = \bar{f}(\bar{x}) + \bar{\nu}_u$$

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Linearization

Choose initial state estimate:

$$\bar{x}_0 = (x_{u0}, y_{u0}, z_{u0}, b_{u0})^T$$

Assume that actual state is given by

$$\bar{x} = \bar{x}_0 + \delta\bar{x}$$

Linearize the pseudorange measurement

$$\bar{\tau} = \bar{f}(\bar{x}) + \bar{\nu}_u \approx \bar{f}(\bar{x}_0) + \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_{\bar{x}_0} (\bar{x} - \bar{x}_0) + \bar{\nu}_u$$

which can be rewritten as

$$\bar{\tau} - \bar{f}(\bar{x}_0) = \delta\tau \approx G_{\bar{x}_0}\,\delta\bar{x} + \bar{\nu}_u$$

and then solved for $\delta\bar{x}$ to find the actual state

Pseudoranging

Thus the linearized pseudorange measurements are

$$\delta\bar{\tau} \approx \frac{\partial f(\bar{x})}{\partial \bar{x}} \delta\bar{x} + \bar{v}_u$$

$$= \begin{bmatrix} \frac{\partial f^{(1)}(\bar{x})}{\partial x_u} & \frac{\partial f^{(1)}(\bar{x})}{\partial y_u} & \frac{\partial f^{(1)}(\bar{x})}{\partial z_u} & \frac{\partial f^{(1)}(\bar{x})}{\partial b_u} \\ \frac{\partial f^{(2)}(\bar{x})}{\partial x_u} & \frac{\partial f^{(2)}(\bar{x})}{\partial y_u} & \frac{\partial f^{(2)}(\bar{x})}{\partial z_u} & \frac{\partial f^{(2)}(\bar{x})}{\partial b_u} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^{(K)}(\bar{x})}{\partial x_u} & \frac{\partial f^{(K)}(\bar{x})}{\partial y_u} & \frac{\partial f^{(K)}(\bar{x})}{\partial z_u} & \frac{\partial f^{(K)}(\bar{x})}{\partial b_u} \end{bmatrix} \delta\bar{x} + \bar{v}_u$$

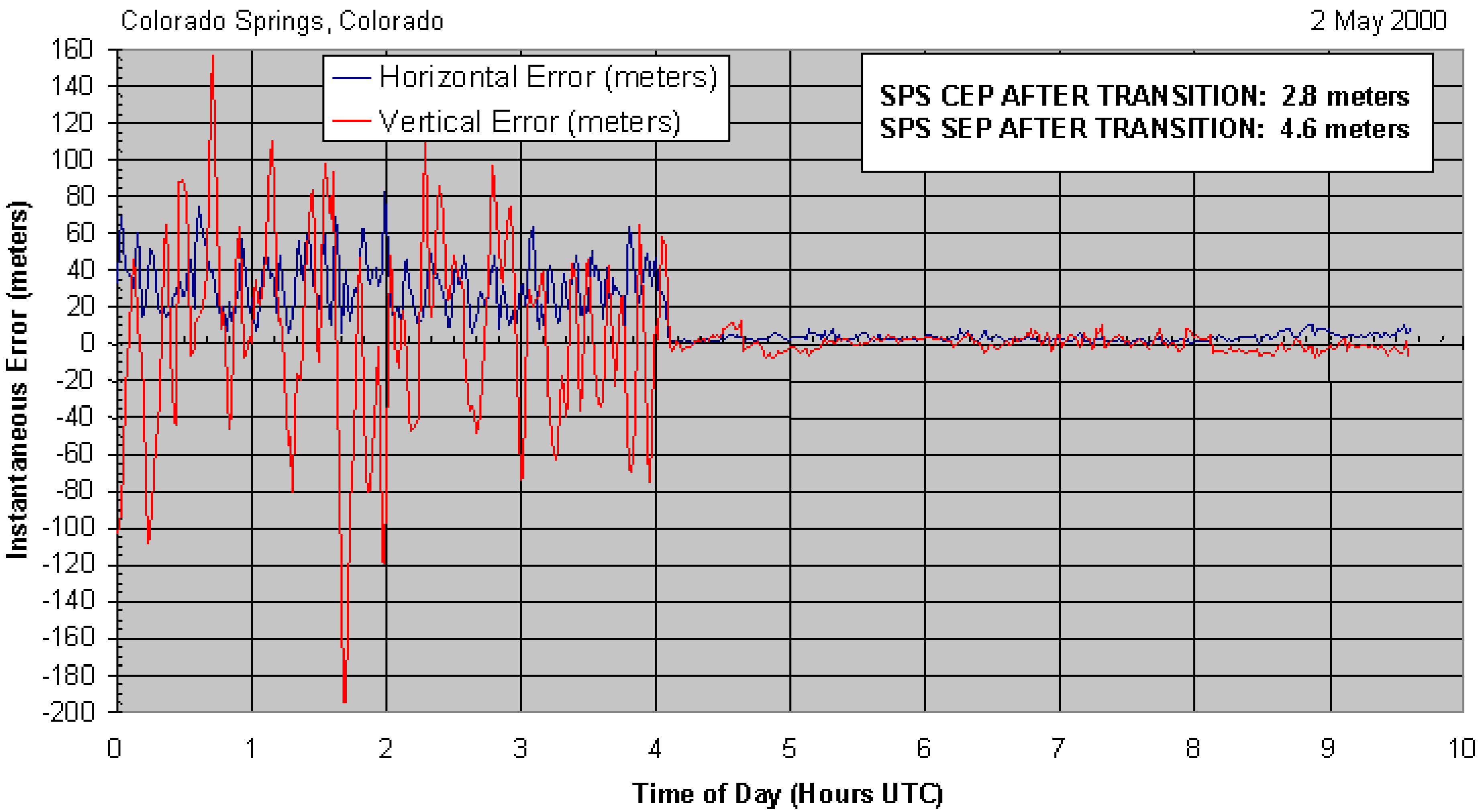
$$\delta\bar{\tau} = G\delta\bar{x} + \bar{v}_u$$

$$\delta\hat{\bar{x}} = (G^T G)^{-1} G^T \delta\bar{\tau}$$

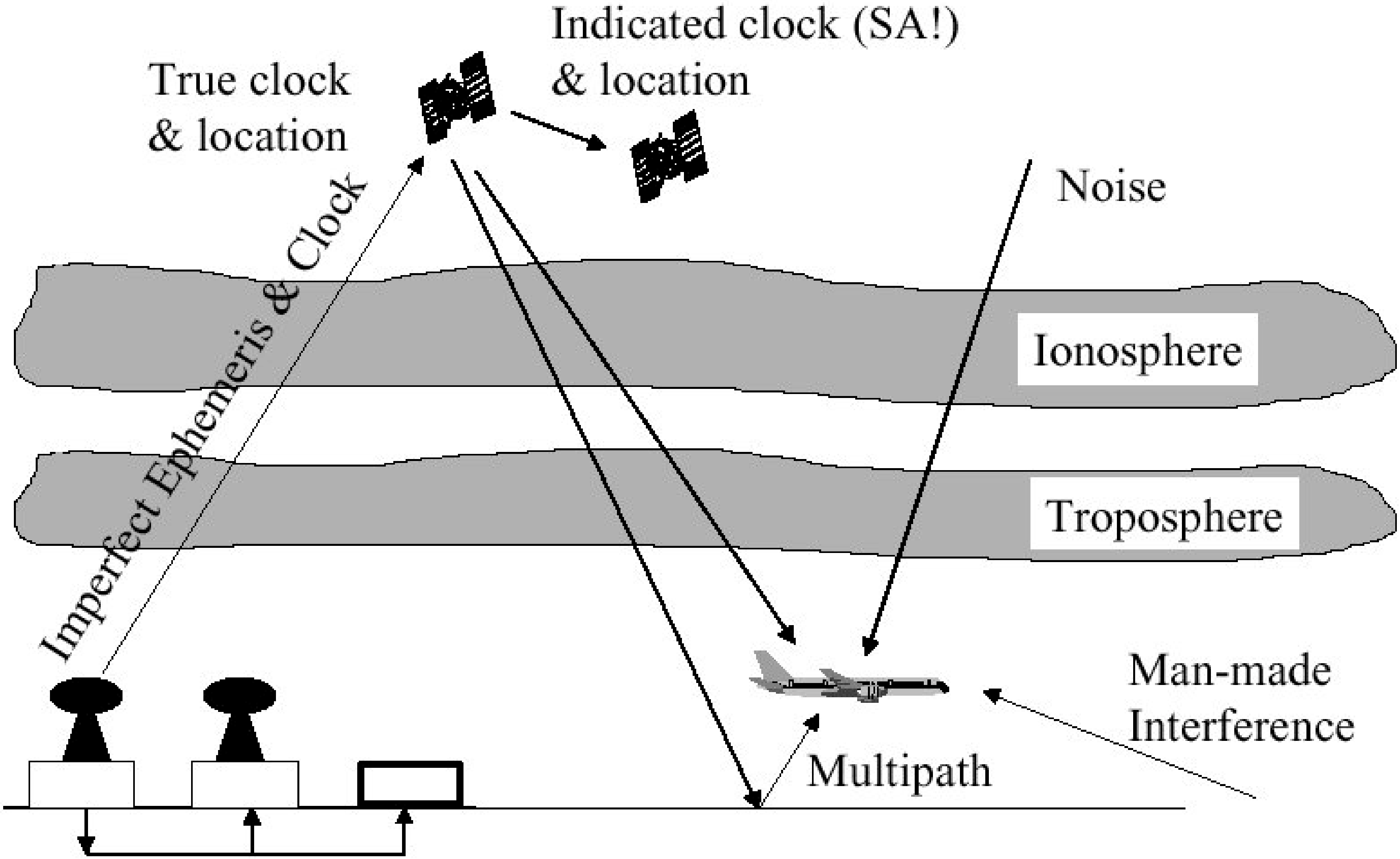
Linearization a function of current best estimate, which is improving, so an iteration might be required.

Depends on how much G changes

SA Transition -- 2 May 2000



Stand Alone GPS Performance



Stand Alone GPS Position Error Budget

Error Source	Bias	Random	Total
Ephemeris	2.1	0.0	2.1
SV Clock	20.0	0.7	20.0
Ionosphere	4.0	0.5	4.0
Troposphere	0.5	0.5	0.7
Multipath	1.0	1.0	1.4
Receiver Noise	0.5	0.2	0.5
PR Error (RMS)	20.5	1.4	20.6
Filtered PR Error	20.5	0.4	20.5
Total Vertical Error VDOP=2.5			51.4 m
Total Horizontal Error HDOP=2.0			41.4 m

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Dilution of Precision (DOP)

- Errors in the previous estimation

$$-_x \quad (G^T G)\;^{1} G^T -_u$$

- Covariance of x is given by:

$$E\left[-_x -_x^T\right] \quad (G^T G)\;^{1} G^T E\left[-_u -_u^T\right] G (G^T G)\;^{1}$$

$$E\left[-_u -_u^T\right] \quad I \;{}^{2}_{v}$$

$$E\left[-_x -_x^T\right] \quad (G^T G)\;^{1} \;{}^{2}_{v}$$

$$\begin{bmatrix} (x\,\mathrm{DOP})^2 & & & \\ & (y\,\mathrm{DOP})^2 & & \\ & & (z\,\mathrm{DOP})^2 & \\ & & & (b\,\mathrm{DOP})^2 \end{bmatrix} {}^{2}_{v}$$

GDOP – "Role of Geometry"

- GDOP (Geometric Dilution of Precision) is then defined as:

$$\text{GDOP} = \sqrt{(x\,\text{DOP})^2 + (y\,\text{DOP})^2 + (z\,\text{DOP})^2 + (b\,\text{DOP})^2}$$

$$= \frac{1}{\sigma_v}\sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + \sigma_b^2}$$

- Note that for dynamic scenarios, $\bar{X}_0$ could be the prior estimate before a measurement update, and $\delta\bar{X}$ gives the modification to the estimate from the new measurement
  - An iteration might be required to update G

Differential - Linearization

Now assume a fixed receiver in a known location:

$$\bar{x}_r = (x_{ur}, y_{ur}, z_{ur}, b_{ur})^T$$

And that the actual state is given by $\bar{x} = \bar{x}_r + \delta\bar{x}$

Linearize the pseudorange measurement

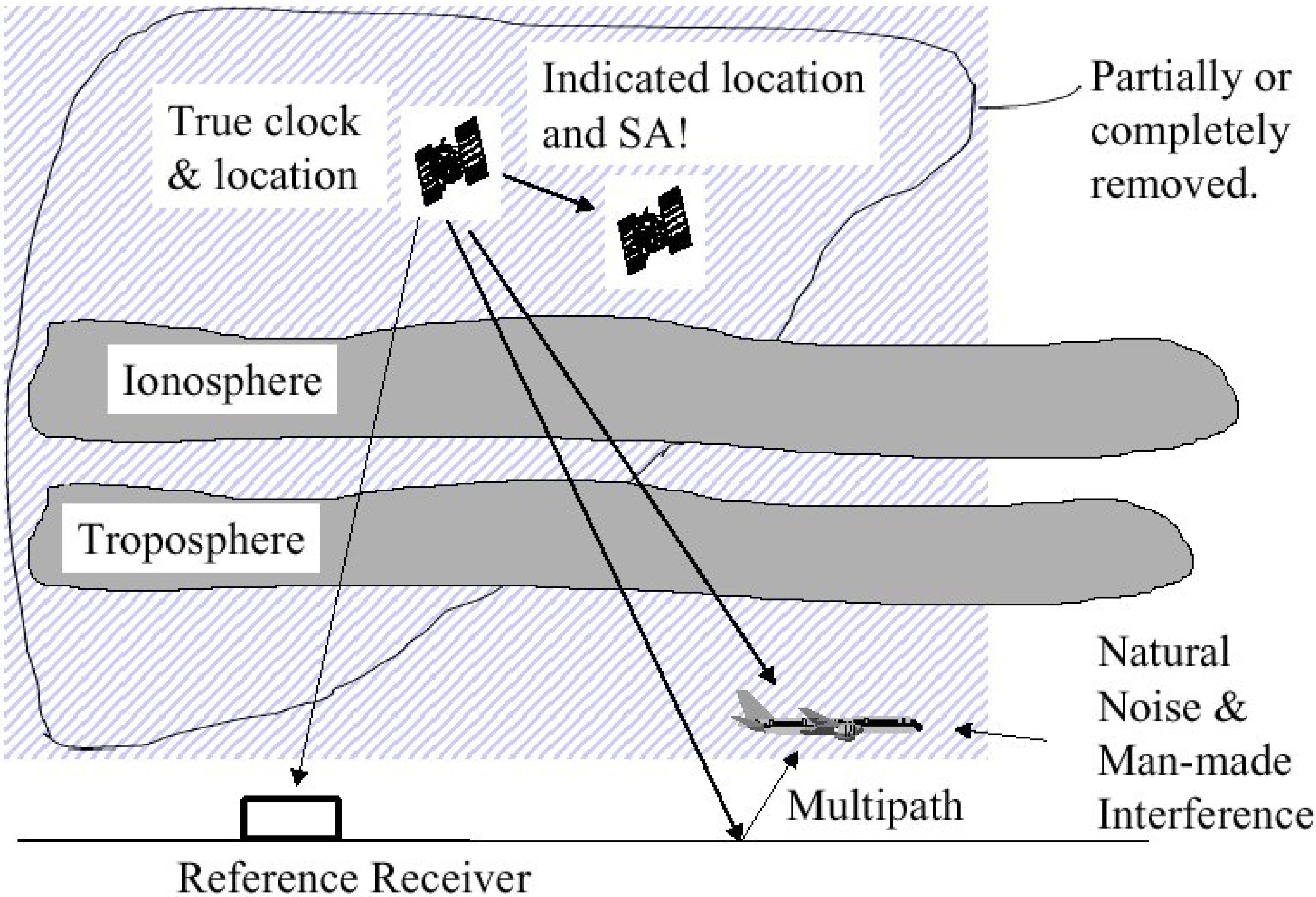
$$\bar{\tau} - \bar{\tau}_r = \bar{f}(\bar{x}) + \bar{\nu}_u - \bar{f}(\bar{x}_r) + \bar{\nu}_u^r$$

$$\delta\tau \approx \bar{f}(\bar{x}_r) + \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_{\bar{x}_r} (\bar{x} - \bar{x}_r) - \bar{f}(\bar{x}_r) + \bar{\nu}_u^D$$

which can be rewritten as $\delta\tau \approx G_{\bar{x}_r}\,\delta\bar{x} + \bar{\nu}_u^D$ and then solved for $\delta\bar{x}$ to find the actual state.

KEY POINT: $\bar{\nu}_u^D$ typically much smaller than $\bar{\nu}_u$

DGPS Errors


True clock
& location
Indicated location
and SA!
Partially or
completely
removed.
Ionosphere
Troposphere
Natural
Noise &
Man-made
Interference
Multipath
Reference Receiver

DGPS Position Error Budget (50 km & prompt corrections)

Error Source	Bias	Random	Total
Ephemeris	0.0	0.0	0.0
SV Clock	0.0	0.7	0.7
Ionosphere	0.0	0.5	0.5
Troposphere	0.0	0.5	0.5
Multipath	1.0	1.0	1.4
Receiver Noise	0.0	0.2	0.2
Reference Errors	0.3	0.2	0.4
PR Error (RMS)	1.0	1.4	1.8
Filtered PR Error	1.0	0.4	1.1
Total Vertical Error VDOP=2.5			2.8 m
Total Horizontal Error HDOP=2.0			2.2 m

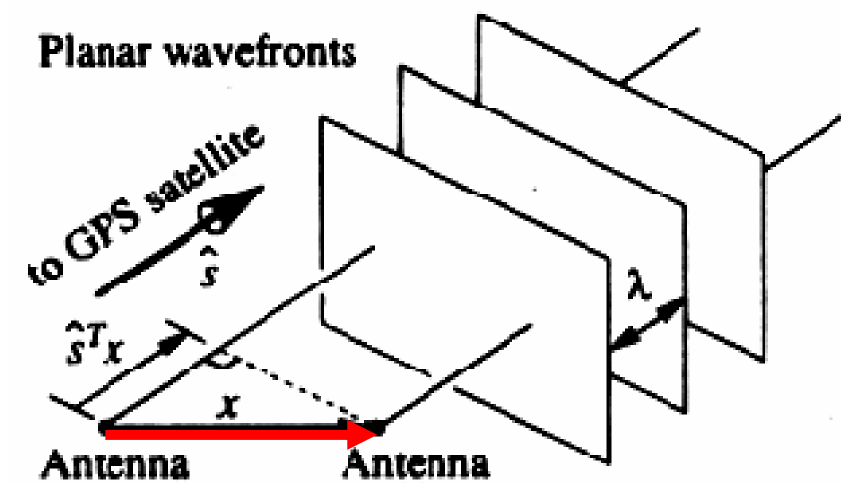
Differential Range with Carrier

- The carrier signal is most often used in a differential mode – 2 antennas (1 receiver)

- Two receivers collect data from the same satellite through separate antennas

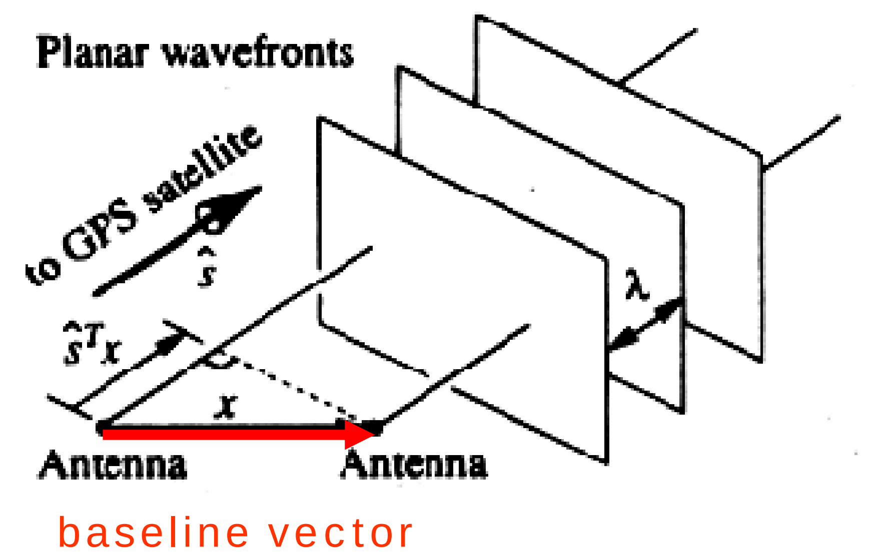
  – Difference the 2 measurements (*differential carrier phase*)

  – DCP is a direct measure of the separation between two antennas projected onto the line-of-sight to the satellite.



S/C Attitude Determination with Differential Carrier Phase

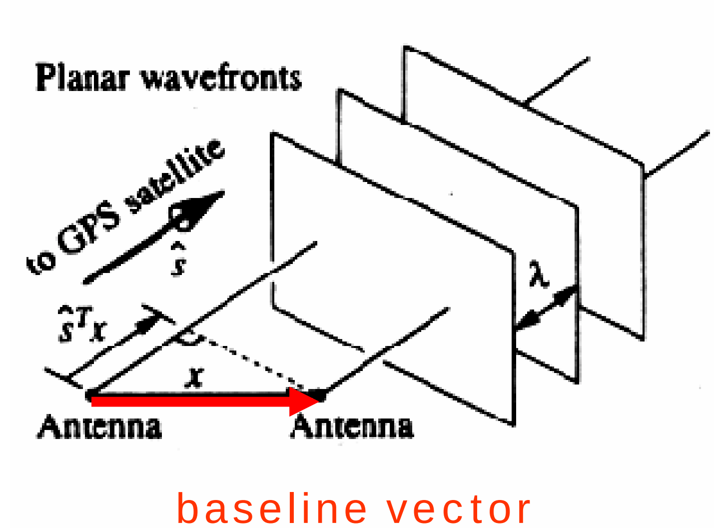
- The carrier signal is most often used in a differential mode – 2 or more antennas.
  - 2 antennas collect data from same satellite.
  - Differentiate two measurements (differential carrier phase)
  - DCP is a direct measure of the baseline vector between two antennas projected onto the line-of-sight to the satellite
    - mm level random noise.



S/C Attitude Determination with Differential Carrier Phase

- With 3 or more DCP measurements, can find out baseline vector x in the GPS frame

$$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \quad \begin{bmatrix} \vec{s}_1^{\,T} \\ \vec{s}_2^{\,T} \\ \vdots \\ \vec{s}_n^{\,T} \end{bmatrix} \vec{x} \quad \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \rightarrow$$

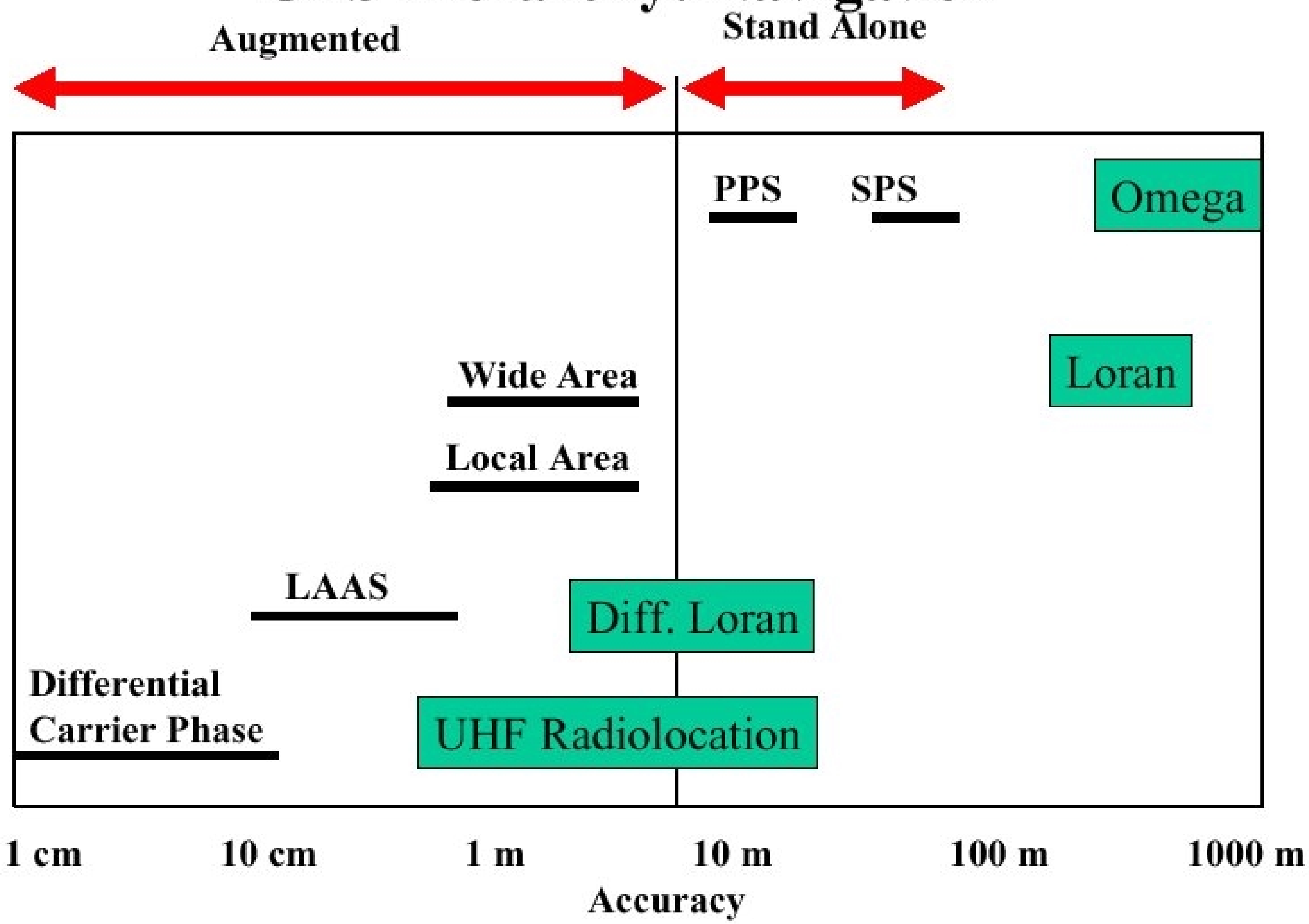


  - Can relate DCP to the relative position of the antennas
    - Attitude extracted from the vector x
  - Measurement ambiguous if baseline longer than wavelength λ
    - But algorithms exist to determine these
  - Then just collect lots of measurements and solve.

Differential Range with Carrier

- Can put antennas on different bodies attached to separate receivers
  - Then can relate DCP to the relative position of the antennas
  - Precise differential range if on separate bodies
- Determining the integers is much harder in this case
  - But algorithms/methods exist to obtain them as well.
- Just collect lots of measurements and solve.

GPS Hierarchy: Navigation



Places to look for more details

- Lightsey, E.G., C.E. Cohen and B.W. Parkinson (1994). "Development of a GPS receiver for reliable real-time attitude determination in space." Proceedings of ION GPS-94, Institute of Navigation, 20-23 Sept., pp. 1677-1684.

- Brock, J.K., R. Fuller, B. Kemper, D. Mleczko, J. Rodden and A. Tadros, "GPS attitude determination and navigation flight experiment," Proceedings of the ION GPS-95, Institute of Navigation, 12-15, Sept., pp. 545-554.

- http://www-ccar.colorado.edu/research/gps/html/gps_attbase.html

- http://sspp.gsfc.nasa.gov/hh/teams/gane.html

- **Bottom line**: GPS has the strong potential to revolution the way that spacecraft attitude and position control is done for future missions, **but**
  - There are no strong market drivers to develop such a receiver, and very few are commercially available.
  - The ones that do exist are **VERY** expensive.